Reduction and approximation in gyrokinetics

Philippe Ghendrih[*], Ricardo Lima[†] and R. Vilela Mendes[‡§¶]

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Abstract

The gyrokinetics formulation of plasmas in strong magnetic fields aims at the elimination of the angle associated with the Larmor rotation of charged particles around the magnetic field lines. In a perturbative treatment or as a time-averaging procedure, gyrokinetics is in general an approximation to the true dynamics. Here we discuss the conditions under which gyrokinetics is either an approximation or an exact operation in the framework of reduction of dynamical systems with symmetry.

1 Introduction

In contrast with the many types of approximation schemes used to deal with complex problems, a *reduction* is an exact process. Reduction is a process by which, given a multidimensional dynamical system, one focus on a small subset of variables, obtaining exact equations for the variables in this subset. Along the way, of course, one looses information about some of the dynamical details of the system. In particular, several distinct evolutions of the whole

[*]Association Euratom-CEA, DRFC/DSM/CEA Cadarache, 13108 St. Paul lez Durance Cédex, France; philippe.ghendrih@cea.fr

[†]Centre de Physique Théorique, CNRS Luminy, case 907, F-13288 Marseille Cedex 9, France; lima@cpt.univ-mrs.fr

[‡]Centro de Fusão Nuclear - EURATOM/IST Association, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

[§]CMAF, Complexo Interdisciplinar, Universidade de Lisboa, Av. Gama Pinto, 2 - 1649-003 Lisboa (Portugal), http://label2.ist.utl.pt/vilela/

[¶]Corresponding author; e-mail: vilela@cii.fc.ul.pt, vilela@cpt.univ-mrs.fr

system may lead to the same dynamical trajectory when projected on the space of the reduced variables but, in any case, one is confident that the dynamics of the reduced variables is exact.

Reduction in mechanics had its origins in the classical works of Euler, Lagrange, Hamilton, Jacobi, Routh and Poincaré. Routh's elimination of cyclic variables and Jacobi's elimination of the node are among the first examples (for reviews of the historical aspects and the several types of reduction we refer to [1] [2] [3]). In general, reduction applies when the system under study possesses some kind of symmetry. Then, the variables associated to the invariance directions are eliminated and one ends up with a dynamical description on a quotient space.

In the context of symplectic manifolds, a construction has now become standard [4] [3] which, for reference and later use, we now summarize:

Let (P, ω) be a symplectic manifold where a Lie group G acts by symplectomorphisms $\left(\phi_g^* \omega = \omega\right)$. An equivariant moment map for the group action is a map $J : P \rightarrow g^*$ (g^* being the dual of the Lie algebra g of G) such that if $\hat{J}$ denotes the dual map from g to the space of smooth functions on P, we have

$$d\left(\hat{J}\left(\xi\right)\right) = i_{\xi_P} \omega \tag{1}$$

and $J\left(\phi_g\left(x\right)\right) = Ad^*_{g^{-1}}\left(J\left(x\right)\right), \forall x \in P, g \in G$. If the momentum map is not equivariant it can be converted into an equivariant one by central extension of the group [5]. Eq.(1) means that each infinitesimal generator ξ_P of g has $\hat{J}\left(\xi\right)$ as an Hamiltonian function.

Let μ be an element of g^* and G_μ its coadjoint isotropy subgroup. For any $G-$invariant Hamiltonian, $J^{-1}\left(\mu\right) \subset P$ is an invariant set for the dynamics and the reduced space $J^{-1}\left(\mu\right)/G_\mu = P_\mu$ is a symplectic manifold with symplectic form Ω_μ determined by $\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$. π_μ is the projection $J^{-1}\left(\mu\right) \rightarrow P_\mu$ and i_μ the inclusion $J^{-1}\left(\mu\right) \rightarrow P$.

In the symplectic reduction the emphasis is on the projection of the Poisson structure and Hamiltonian dynamics to a quotient space by the action of a symmetry group. In the Lagrangian approach, reduction follows a different approach. Reduced variables are identified and then one proceeds to carry the variational structure to a quotient space. In the process not only the completeness of the reduced variables set should be checked, but also whether the variational structure can be carried over to the quotient space. Here, in principle, a symmetry group need not to be known to begin with. One may

start by conjecturing some tentative set of reduced variables (functions of the original variables). Then, using the Poisson brackets of the original variables, check whether this trial set is algebraically closed. If not, the Poisson brackets themselves will suggest new variables to close the set. Then, to be sure that the variational structure is carried over to the quotient space, it is necessary to check whether the (exact) dynamics of the reduced variables may be obtained from a Lagrangian written on these variables alone. If that is possible, an exact reduction has been performed.

Even when an exact reduction on some subset of variables is not possible, the reduction point of view provides a step by step approach to an approximation scheme. For example, it is possible that the trial set of reduced variables is algebraically closed with a consistent bracket structure, but that their dynamics cannot be entirely defined by an action principle containing only these variables. To proceed it must be necessary at this step to propose some kind of approximation, but at least the method makes it clear how much of the reduction process is exact and how much it implies an approximation.

The same methodology of approximation through reduction may be used in the group theoretical setting. Suppose that some dynamical process is known that is not exactly symmetric, but we are interested only on the dynamical features consistent with the symmetry. Then an *approximation-through-reduction* may be obtained by projecting the dynamics on the space of functions that possess the required symmetry.

In the context of plasma physics in strong magnetic fields, two very different time scales rule the physical phenomena. One is the fast time scale of gyromotion of the ions around the magnetic field (of order $\frac{2\pi mc}{eB}$) and the other the longer time scale associated to the electric and magnetic gradient and curvature drifts. An important simplification arises when the two time scales can be treated separately. This might not be possible, for example in the study of fluctuation phenomena. However, for some stability and transport problems it is indeed an useful approximation. When the fast time scale is separated, or integrated over, one obtains the so called gyrokinetic and gyrocenter equations. The usual way this is done is either by simple averaging the single particle dynamics over the gyroperiod [6] [7] or, more accurately, by performing a Lie-transform perturbative expansion to obtain an action two-form where the gyroangle dependence is to be asymptotically eliminated. In both cases the point of view is to address the gyrokinetics formulation as an approximation to the plasma dynamics. The Lie-transform

perturbative approach has been extensively developed [8] [9] [10] [11] leading, for example, to a third-order perturbative analysis of a plasma moving with a nonuniform fluid velocity [12].

Here, our purpose is to find out how much of the gyrokinetics program may be framed as an exact reduction and how much it is an approximation. In Sect.2 we deal with Lagrangian reduction as a time-averaging process, following a procedure quite analogous to the one that leads from the Lagrangian dynamics of many particles to the Euler equation. The only difference is that now we define time-averaged Eulerian coordinates.

The Lie-transform approach to gyrokinetics ([13] and references therein) has been extensively studied and provides successive approximations of practical interest to the reduced Vlasov and Maxwell equations. Its purpose is to obtain the asymptotic elimination of the gyroangle dependence. However to show that this procedure provides an exact reduction of the dynamics would require the proof of convergence of the perturbative series, what has not yet been done. In Sect.3, following a non-perturbative approach, we show how gyrokinetics may be considered an exact reduction of the dynamics in the sense of the symplectic reduction scheme discussed above [4].

2 Lagrangian time-averaging reduction

Here we discuss, gyrokinetics reduction as a time-averaging process, in particular to exhibit the meaning and shortcomings of such an approach. We start from the non-relativistic particles plus electromagnetic fields Lagrangian [14]

$$\begin{aligned} L &= \int dz_0 f\left(z_0\right)\left[\frac{m}{2}\overset{\bullet}{r}^2\left(z_0,t\right) - e\int dx\left(\Phi\left(x,t\right) - \frac{\overset{\bullet}{r}\left(z_0,t\right)}{c}\bullet A\left(x,t\right)\right)\delta\left(x-r\left(z_0,t\right)\right)\right] \\ &\quad +\frac{1}{8\pi}\int dx\left(\left|E\left(x,t\right)\right|^2 - \left|B\left(x,t\right)\right|^2\right) \end{aligned} \tag{2}$$

$z_0 = (r_0, p_0)$ stands for a particle coordinate in phase space, $r(z_0, t)$ is the position of the particle that at time zero was at z_0, and $f(z_0)$ is the particle density at time zero, that is

$$f\left(z_0\right) = \sum_{i=1}^{N}\delta\left(r_0 - r_0^{(i)}\right)\delta\left(p_0 - p_0^{(i)}\right) \tag{3}$$

for N particles. We consider only one ion species with mass m and charge e. Generalization of all statements to an ensemble of different ion species is straightforward.

The Lagrangian (2) contains information about the identity and dynamics of each particle at all times. We now define our reduced variables

$$f_h(R,P,t) = \int dz_0 f(z_0)\, \delta(R - \overline{r}(z_0,t))\, \delta\left(P - \frac{\overline{\pi}(z_0,t)}{f(z_0)}\right) \tag{4}$$

where

$$\begin{aligned} \overline{r}(z_0,t) &= \frac{1}{\lambda_t}\int d\eta h_t(\eta)\, r(z_0, t-\eta) \\ \overline{\pi}(z_0,t) &= \frac{1}{\lambda_t}\int d\eta h_t(\eta)\, \pi(z_0, t-\eta) \end{aligned} \tag{5}$$

The kernel $h_t(\eta)$ is a function equal to one in the interval $\left[-\frac{\lambda_t}{2}, \frac{\lambda_t}{2}\right]$ and zero otherwise. The reason for the division by $f(z_0)$ in the second delta function of Eq.(4) comes from the fact that the canonical momentum is

$$\pi(z_0,t) = \frac{\delta L}{\delta \overset{\bullet}{r}(z_0,t)} = f(z_0)\left(m \overset{\bullet}{r}(z_0,t) + \frac{e}{c} A(r(z_0,t))\right) \tag{6}$$

In the new variables two types of information are lost. First, the particle identities are lost, being replaced by densities at points in phase space. Second, the dynamics itself is averaged around each time t with a window of size λ_t, that need not be the same for all t.

The next step is to check the consistency of this reduction. First one checks whether the new set $\{f_h(R,P,t)\}$ is algebraically closed and whether its Poisson bracket may be expressed on the variables (R,P). We denote $Z=(R,P)$ and compute the Poisson bracket of the $f^{'}s$ in the Lagrangian variables (r,π).

$$\begin{aligned}
&\left\{f_h(Z,t)\, f_h(Z',t)\right\}_{(r,\pi)} = \\
&= \tfrac{1}{\lambda_t^2}\int dz_0 f(z_0)\Big[\delta'(\overline{r}(z_0,t)-R)\,\delta\left(\tfrac{\overline{\pi}}{f(z_0)}-P\right)\delta\left(\overline{r}-R'\right)\delta'\left(\tfrac{\overline{\pi}}{f(z_0)}-P'\right) \\
&\qquad\qquad -\delta(\overline{r}(z_0,t)-R)\,\delta'\left(\tfrac{\overline{\pi}}{f(z_0)}-P\right)\delta'\left(\overline{r}-R'\right)\delta\left(\tfrac{\overline{\pi}}{f(z_0)}-P'\right)\Big] \\
&= \tfrac{1}{\lambda_t^2}\left[f_h(R',P,t)\,\delta'(R'-R)\,\delta'(P-P') - f_h(R,P',t)\,\delta'(R-R')\,\delta'(P'-P)\right] \\
&= \tfrac{1}{\lambda_t^2}\int dZ'' f_h(Z'',t)\left\{\delta(Z-Z''),\delta(Z'-Z'')\right\}_{(R,P)}
\end{aligned} \tag{7}$$

One sees that not only is the set $\{f_h(R,P,t)\}$ algebraically closed, but also its Poisson bracket may be written in the variables (R,P). The conclusion is that this set of (Eulerian) time-averaged densities is a kinematically consistent reduction. The quantities R and P are in some sense coordinates in a time-averaged phase-space. However, one should remember that, from the reduction point of view it is the $f^{'}s$ that are the actual dynamical variables which, with its bracket and reduced Hamiltonian, control the dynamics. The set (R,P) is just a set of labels for the dynamical variables.

The next step in the reduction process is to check the dynamical law and, in particular, whether the dynamics may be completely expressed in the new variables. The Hamiltonian that follows from (2) by the Legendre transform is

$$\begin{aligned} H &= \int dz_0 f(z_0)\left\{\frac{1}{2m}\left|\frac{\pi(z_0,t)}{f(z_0)}-\frac{e}{c}A(r(z_0,t))\right|^2+e\Phi(r(z_0,t))\right\} \\ &\quad +\frac{1}{8\pi}\int dx\left(|E(x,t)|^2+|B(x,t)|^2\right) \end{aligned} \tag{8}$$

Then we compute the Poisson bracket in the original Lagrangian coordinates. Averaging the computation over the interval $\tau\in[t-\frac{\lambda_t}{2},t+\frac{\lambda_t}{2}]$, considering at each τ a complete set of canonical variables $(r(z_0,\tau),\pi(z_0,\tau))$.

$$\begin{aligned} &\overset{\bullet}{f_h}(Z,t)=\{f_h(Z,t),H\}_{(r,\pi)}=\tfrac{1}{\lambda_t}\int d\tau\,\{f_h(Z,t),H(\tau)\}_{(r(z_0,\tau),\pi(z_0,\tau))} \\ &=\tfrac{1}{\lambda_t^2}\int d\tau\int dz_0 f(z_0)\left[\delta'(\overline{r}(z_0,t)-R)\,\delta\left(\tfrac{\overline{\pi}(z_0,t)}{f(z_0)}-P\right)\cdot\tfrac{1}{m}\left(\tfrac{\pi(z_0,\tau)}{f(z_0)}-\tfrac{e}{c}A(z_0,\tau)\right)\right. \\ &\qquad\left.+\,\delta(\overline{r}(z_0,t)-R)\,\delta'\left(\tfrac{\overline{\pi}(z_0,t)}{f(z_0)}-P\right)\cdot\left(\tfrac{e}{mc}\left(\tfrac{\pi(z_0,\tau)}{f(z_0)}-\tfrac{e}{c}A(z_0,\tau)\right)\cdot\nabla A-e\nabla\phi\right)\right] \end{aligned} \tag{9}$$

We see that, by taking averages over τ and using the multiplication by the delta function in the first term, one obtains the evolution equation in the renormalized time $\frac{t}{\lambda_t}$

$$\frac{\partial f_h(Z,t)}{\partial\left(\frac{t}{\lambda_t}\right)}+\left(\frac{P}{m}-\frac{e}{mc}\overline{A}\right)\cdot\frac{\partial f_h(Z,t)}{\partial R}+\overline{\left(\frac{e}{mc}\left(\frac{\pi}{f(z_0)}-\frac{e}{c}A\right)\cdot\nabla A-e\nabla\phi\right)}\cdot\frac{\partial f_h(Z,t)}{\partial P}=0 \tag{10}$$

In this equation the coefficient of the $\frac{\partial}{\partial R}$ derivative takes the usual form for a Vlasov equation in time-averaged variables but, unless the average of the products coincide with the product of the averages, that is not the case for

the coefficient of the $\frac{\partial}{\partial P}$ term. For the reduction to be exact one should be able to express the dynamics of the reduced densities $f_h(Z,t)$ purely in terms of the variables (R,P) and the time-averaged field potentials $\left(\overline{\phi},\overline{A}\right)$. It is at this step that the *reduction* may become an *approximation*. However, if it is possible, by an adequate choice of the time-dependent averaging interval λ_t, to insure that the average of the products in (10) may be written in the averaged variables, then one has an exact reduction. Otherwise the reduced dynamics implies an approximation.

As an example of a particular situation where time-averaging provides an exact reduction consider constant electromagnetic fields with the uniform magnetic field B oriented along the $z-$axis. The particle equations of motion are

$$\begin{array}{rcl} m\frac{dv_z}{dt} & = & eE_z \\ m\frac{dv_x}{dt} & = & eE_x + \frac{e}{c}v_y B \\ m\frac{dv_y}{dt} & = & eE_y - \frac{e}{c}v_x B \end{array} \tag{11}$$

Then

$$\begin{array}{rcl} v_x & = & \frac{c}{B}\left(1-\cos\left(\omega t\right)\right)E_y + \frac{1}{B}\left(cE_x + v_y\left(0\right)B\right)\sin\left(\omega t\right) \\ & & +v_x\left(0\right)\cos\left(\omega t\right) \\ v_y & = & -\frac{c}{B}\left(1-\cos\left(\omega t\right)\right)E_x + \frac{1}{B}\left(cE_y - v_x\left(0\right)B\right)\sin\left(\omega t\right) \\ & & +v_y\left(0\right)\cos\left(\omega t\right) \\ v_z & = & \frac{e}{m}tE_z + v_z\left(0\right) \end{array} \tag{12}$$

$\omega = \frac{e|B|}{mc}$. Choosing $\lambda = \frac{2\pi}{\omega}$, the corresponding averaged dynamics is

$$\begin{array}{rcl} \overline{v}_x & = & c\frac{\overline{E}_y}{B} \\ \overline{v}_y & = & -c\frac{\overline{E}_x}{B} \\ \overline{v}_z & = & \frac{e}{m}t\overline{E}_z + \overline{v}_z\left(0\right) \\ \frac{d\overline{v}_x}{dt} & = & 0 \\ \frac{d\overline{v}_y}{dt} & = & 0 \\ \frac{d\overline{v}_z}{dt} & = & \frac{e}{m}\overline{E}_z \end{array} \tag{13}$$

and the reduced Vlasov equation

$$\frac{\partial f}{\partial t} + \overline{v}_x\frac{\partial f}{\partial \overline{x}} + \overline{v}_y\frac{\partial f}{\partial \overline{y}} + \overline{v}_z\frac{\partial f}{\partial \overline{z}} + \frac{e}{m}\overline{E}_z\left(f\right)\frac{\partial f}{\partial \overline{v}_z} = 0 \tag{14}$$

As expected in an exact reduction, the angle, argument of the trigonometric functions, being an ignorable coordinate for the averaged dynamics, the

number of relevant coordinates is reduced from six, (x, y, z, v_x, v_y, v_z), to four, $(\overline{x}, \overline{y}, \overline{z}, \overline{v}_z)$.

Here we have analyzed reduction as a time-averaging process to emphasize its different aspects, namely the structural (or kinematical) level that concerns the Poisson brackets and the dynamical level that refers to the equations of motion. Although we have considered an averaging interval λ_t that depends on time, it must be pointed out that, for general magnetic field configurations, the gyrokinetics approximation is more general in the sense that the averaging depends not only in time but also on the position of the particles. In the present Lagrangian reduction scheme it would correspond to having a kernel $h = h(z_0, t)$.

For general electromagnetic fields, the difficulties with the time-averaging point of view lead naturally to the more appropriate notion of gyroangle independence, which is also the point of view of the Lie-transform perturbative approach [13]. Here we also follow the gyroangle independence point of view but, instead of a perturbative approach, we prove the existence of an exact invariant from which an exact reduction is shown to follow.

3 Gyrokinetics as an exact reduction

In the extended phase-space $\left(\vec{x}, \vec{p}, t, h\right)$, the Hamiltonian of a particle moving in an electromagnetic field is [15]

$$H\left(\vec{x}, \vec{p}, t, h\right) = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\Phi - h \tag{15}$$

the non-vanishing elements of the Poisson tensor being

$$\begin{array}{lcl} \{x^i, p^j\} & = & \delta^{ij} \\ \{t, h\} & = & -1 \end{array} \tag{16}$$

Changing coordinates to $\left(\vec{x}, \vec{v}, t, k\right)$ with

$$\begin{array}{lcl} \vec{v} & = & \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right) \\ k & = & h - e\Phi \end{array} \tag{17}$$

leads to

$$H\left(\vec{x}, \vec{v}, t, k\right) = \frac{1}{2}m\,\vec{v}^{\,2} - k \tag{18}$$

and

$$\begin{array}{rcl} \sigma^{iv^j} = \{x^i, v^j\} & = & \frac{1}{m}\delta^{ij} \\ \sigma^{tk} = \{t, k\} & = & -1 \\ \sigma^{v^i v^j} = \{v^i, v^j\} & = & \frac{e}{m^2 c} B^{ij} \\ \sigma^{v^i k} = \{v^i, k\} & = & -\frac{e}{m} E^i \end{array} \tag{19}$$

$B^{ij} = \epsilon^{ijk} B_k$

It is convenient to decompose the velocity into magnetic field adapted components,

$$\begin{array}{rcl} v_\parallel & = & v_i \hat{b}_i \\ \vec{v}_\perp & = & \vec{v} - \hat{b}\left(\vec{v} \cdot \hat{b}\right) \end{array} \tag{20}$$

$\hat{b} = \frac{\vec{B}}{\left|\vec{B}\right|}$, for which the equations of motion are obtained from (18) and (19) by $\frac{dF}{dt} = \{F, H\}$

$$\begin{array}{rcl} \frac{d}{dt} v_\parallel & = & \frac{e}{m} E_\parallel + \vec{v}_\perp \cdot \left(\vec{v} \cdot \nabla\right) \hat{b} \\ \frac{d}{dt} \vec{v}_\perp & = & \frac{e}{m} \vec{E}_\perp + \frac{e}{mc}\left(\vec{v}_\perp \times \vec{B}\right) - v^i \left(\vec{v} \cdot \nabla\right)\left(\hat{b}^i \hat{b}\right) \end{array} \tag{21}$$

For bounded electromagnetic fields with bounded derivatives, the right-hand side of the system of equations (21) is locally lipschitzian. This insures existence and uniqueness of the solution for a time interval.

To prove the existence of an exact reduction of a dynamical system one has to identify a symmetry group or, equivalently, the existence of one or more invariants. Our method depends on the construction of a formal invariant. Existence of such invariants, to all orders in perturbation theory, for charged particle motion in a strong magnetic field was first pointed out by Kruskal[17]. Here we attempt an explicit construction of an exact invariant. The technique hinges on transforming the equation for the transverse velocity to the form

$$\frac{d}{dt}\left(\vec{v}_\perp - \vec{u}_\perp\right) = k\left(\vec{v}_\perp - \vec{u}_\perp\right) \times \vec{B} - \hat{b}\,\Gamma\left(\vec{x}, \vec{v}, t\right) - \alpha\left(\vec{x}, \vec{v}, t\right)\left(\vec{v}_\perp - \vec{u}_\perp\right) \tag{22}$$

When this is achieved it is easy to see that

$$M = \frac{\left|\vec{v}_\perp - \vec{u}_\perp\right|^2}{F} \tag{23}$$

is a constant of motion provided

$$F=\exp\left(-2\int^{t}\alpha\left(\overrightarrow{x},\overrightarrow{v},\tau\right)d\tau\right) \tag{24}$$

The results are summarized in the following proposition:

In the domain of existence of bounded solutions of the system (21) and for bounded and sufficiently smooth electromagnetic fields, there are functions $\Gamma\left(\overrightarrow{x},\overrightarrow{v},t\right)$, $\alpha\left(\overrightarrow{x},\overrightarrow{v},t\right)$ *and a transversal vector function* $\overrightarrow{u}_{\perp}\left(\overrightarrow{x},\overrightarrow{v},t\right)$ *such that the equation (22) holds. For fields that have no explicit time dependence and sufficiently large magnetic field we also have* $\overrightarrow{u}_{\perp}=\overrightarrow{u}_{\perp}\left(\overrightarrow{x},\overrightarrow{v}\right)$, $\alpha=\alpha\left(\overrightarrow{x},\overrightarrow{v}\right)$ *and* $\Gamma=\Gamma\left(\overrightarrow{x},\overrightarrow{v}\right)$.

Proof : Without explicit time-independence of $\overrightarrow{u}_{\perp}$, α and Γ, the result is fairly trivial. It suffices to rewrite (22) as

$$\frac{d}{dt}\overrightarrow{u}_{\perp}-k\,\overrightarrow{u}_{\perp}\times\overrightarrow{B}+\alpha\left(\overrightarrow{x},\overrightarrow{v},t\right)\overrightarrow{u}_{\perp}=\left(\frac{d}{dt}+\alpha\left(\overrightarrow{x},\overrightarrow{v},t\right)-k\,\overrightarrow{B}\times\right)\overrightarrow{v}_{\perp}+\hat{b}\,\Gamma\left(\overrightarrow{x},\overrightarrow{v},t\right) \tag{25}$$

and then, the assumed boundedness of the fields and the solutions of (21) implies the existence of a solution $\overrightarrow{u}_{\perp}\left(\overrightarrow{x},\overrightarrow{v},t\right)$ for each choice of $\Gamma\left(\overrightarrow{x},\overrightarrow{v},t\right)$ and $\alpha\left(\overrightarrow{x},\overrightarrow{v},t\right)$.

Less trivial is to show the existence of time-independent solutions. Because we are going to construct the quantities $\overrightarrow{u}_{\perp}$, α and Γ and the invariant as a formal operator series, it is important to control the magnitude of the operator action in the space of velocities, in particular to guarantee that it does not grow with $|B|$, the magnetic field intensity. In the second equation in (21) we bring the term $\frac{e}{mc}\left(\overrightarrow{v}_{\perp}\times\overrightarrow{B}\right)$ to the left-hand side

$$\left(\tfrac{d}{dt}+\tfrac{e}{mc}\overrightarrow{B}\times\right)\overrightarrow{v}_{\perp}=\tfrac{e}{m}\overrightarrow{E}_{\perp}-v^{i}\left(\overrightarrow{v}\cdot\nabla\right)\left(\hat{b}^{i}\hat{b}\right) \tag{26}$$

and rewrite the operator $\left(\frac{d}{dt}+\frac{e}{mc}\overrightarrow{B}\times\right)$ as a derivation

$$D=\frac{d}{dt}+\frac{e\left|B\right|}{mc}\left(v_{\perp}^{(1)}\frac{\partial}{\partial v_{\perp}^{(2)}}-v_{\perp}^{(2)}\frac{\partial}{\partial v_{\perp}^{(1)}}\right) \tag{27}$$

$v_\perp^{(1)}$ and $v_\perp^{(1)}$ being the components of $\vec{v}_\perp$ in an arbitrary transverse coordinate system. Then the equations (21) become

$$\begin{aligned} Dv_\parallel &= \tfrac{e}{m}E_\parallel + \vec{v}_\perp \cdot \left(\vec{v}\cdot\nabla\right)\hat{b} \\ D\,\vec{v}_\perp &= \tfrac{e}{m}\,\vec{E}_\perp - v^i\left(\vec{v}\cdot\nabla\right)\left(\hat{b}^i\,\hat{b}\right) \end{aligned} \tag{28}$$

We see that the right-hand side no longer involves explicit dependence on the ,magnetic field, that is, the action of the D operator does not introduce frequencies comparable to the Larmor frequency. It is the choice of the D operator for the operator expansions that, for large B, implements the separation of time scales.

We now apply the identity,

$$\vec{a}_\perp = -\frac{1}{\left|\vec{B}\right|^2}\left(\vec{a}_\perp \times \vec{B}\right)\times\vec{B} \tag{29}$$

that holds for transversal fields, to the first and the transversal part of the second term in the right-hand side of the second equation in (28), to obtain

$$\begin{aligned} D\left(\vec{v}_\perp - c\tfrac{\vec{E}_\perp\times\vec{B}}{|B|^2} + \tfrac{mcv_\shortparallel}{e|B|^2}\left(\vec{v}\cdot\nabla\right)\hat{b}\times\vec{B}\right) &= \tfrac{e}{mc}\left(-c\tfrac{\vec{E}_\perp\times\vec{B}}{|B|^2} + \tfrac{mcv_\shortparallel}{e|B|^2}\left(\vec{v}\cdot\nabla\right)\hat{b}\times\vec{B}\right)\times\vec{B} \\ &\quad - D\left\{c\tfrac{\vec{E}_\perp\times\vec{B}}{|B|^2} - \tfrac{mcv_\shortparallel}{e|B|^2}\left(\vec{v}\cdot\nabla\right)\hat{b}\times\vec{B}\right\} - \hat{b}\,v^i\left(\vec{v}\cdot\nabla\right)\hat{b}^i \end{aligned} \tag{30}$$

Separating the transversal and longitudinal components of the term $D\{\cdots\}$ in the right-hand side of Eq.(30), using the identity (29) for the transversal component and iterating the process one finally obtains

$$\frac{d}{dt}\left(\vec{v}_\perp - \vec{u}_\perp\right) = \left(\frac{e}{mc}+\gamma\right)\left(\vec{v}_\perp - \vec{u}_\perp\right)\times\vec{B} - \hat{b}\,\Gamma - \alpha\left(\vec{v}_\perp - \vec{u}_\perp\right) \tag{31}$$

with $\vec{u}_\perp$, Γ and α given by

$$\vec{u}_\perp = \left\{1 - \tfrac{mc}{e|B|^2}\,\vec{B}\times D\right\}^{-1}\left(c\tfrac{\vec{E}_\perp\times\vec{B}}{|B|^2} - \tfrac{mcv_\shortparallel}{e|B|^2}\left(\vec{v}\cdot\nabla\right)\hat{b}\times\vec{B}\right) \tag{32}$$

and

$$\begin{aligned} \Gamma &= \hat{b}\cdot D\left(1 - \tfrac{mc}{e|B|^2}\,\vec{B}\times D\right)^{-1}\left(c\tfrac{\vec{E}_\perp\times\vec{B}}{|B|^2} - \tfrac{mcv_\shortparallel}{e|B|^2}\left(\vec{v}\cdot\nabla\right)\hat{b}\times\vec{B}\right) \\ &\qquad + \hat{b}\cdot\chi + v^i\left(v\cdot\nabla\right)\hat{b}^i \end{aligned} \tag{33}$$

$$\alpha = \frac{\chi \cdot \left(\vec{v}_\perp - \vec{u}_\perp\right)}{\left|\vec{v}_\perp - \vec{u}_\perp\right|^2} \tag{34}$$

and

$$\chi = \left(\frac{d}{dt} - D\right) \vec{u}_\perp \tag{35}$$

$$\gamma = -\left(\frac{\left(\vec{v}_\perp - \vec{u}_\perp\right)}{\left|\vec{v}_\perp - \vec{u}_\perp\right|^2} \times \frac{\vec{B}}{|B|^2}\right) \cdot \chi \tag{36}$$

Notice that $\frac{d}{dt} = \frac{\partial}{\partial t} + \left(\vec{v} \cdot \nabla\right)$. When D is applied to the velocities it is understood that it is a replacement by the right-hand side of Eqs.(28). When D is applied to the fields, explicit time dependence in $\vec{u}_\perp$ and Γ occurs only if the fields themselves have an explicit time dependence. Therefore, whenever a proper meaning is given to the series in (32) and the fields are not explicitly dependent on time, one proves the existence of time-independent solutions $\vec{u}_\perp = \vec{u}_\perp \left(\vec{x}, \vec{v}\right)$, $\Gamma = \Gamma\left(\vec{x}, \vec{v}\right)$ and $\alpha\left(\vec{x}, \vec{v}\right)$.

We now discuss the convergence of the formal series in (32). The operator $\vec{B} \times D$ having the real line as spectrum, $1 - \frac{mc}{e|B|^2} \vec{B} \times D$ cannot have an inverse defined in the whole L^2. Instead we consider the action of the operator at sucessively higher orders on a space of velocities and fields bounded by some quantity M. Because of the choice of the operator D, the terms do not grow with B, however, because of the nonlinear nature of the (28) action, there is a proliferation of terms and, at most, we obtain a bound

$$\left|\vec{u}_\perp\right| \leq \sum_n n! \left(\frac{M}{|B|}\right)^n$$

which does not insure convergence. However, multiplying and dividing by $n!$ and exchanging the order of sum and integral one obtains

$$\left|\vec{u}_\perp\right| \leq \int_d \sum_n \left(\frac{Mz}{|B|}\right)^n e^{-z} dz$$

where we have used the integral representation

$$n! = \int_d dz z^n e^{-z}$$

d denoting the integration along the line $(1+\varepsilon)\,\eta$ (η real $\in [0,\infty)$) in the complex plane. Then

$$\left|\overrightarrow{u}_{\perp}\right| \leq C \int_{d} \sum_{n} \left(1 - \frac{Mz}{|B|}\right)^{-1} e^{-z} dz$$

meaning that $\left|\overrightarrow{u}_{\perp}\right|$ is bounded by a series that is Borel summable along a direction not containing the real half-line. Then the formal series in (32) is also expected to be Borel summable. In this case, by a theorem of Borel [18] [19], one solution $\overrightarrow{\tilde{u}}_{\perp}$ exists for which (32) is an asymptotic series. The nature of (32) as an asymptotic series has the implication that for practical calculations the number of terms to be kept depends on both the magnitudes of the velocities and the magnetic field.

Now from the existence of a vector function $\overrightarrow{u}_{\perp}$, satisfying (22), it follows that for M, constructed in (23), $\frac{d}{dt}M = 0$. Therefore, an exact reduction is now possible following the Marsden-Weinstein theory [4]. M itself is the (dual) moment map (the $\hat{J}(\xi)$ in Eq.1) that by

$$\frac{dQ}{dt} = \{Q, M\} \tag{37}$$

generates the action of the symmetry group on phase-space functions $Q\left(\overrightarrow{x}, \overrightarrow{v}\right)$.

For each value μ of the invariant M one obtains a symplectic reduced space of dimension four. Existence of a set $\left(\overrightarrow{Q}, Q_v\right)$ of four $Q-$coordinates in the reduced space follows from the existence of solutions for the linear (in Q) equation

$$\{Q, M\} = 0 \tag{38}$$

that is,

$$\frac{\partial M}{\partial v^i}\sigma^{v^i j}\frac{\partial Q}{\partial x^j} + \frac{\partial M}{\partial x^i}\sigma^{i v^j}\frac{\partial Q}{\partial v^j} + \frac{\partial M}{\partial v^i}\sigma^{v^i v^j}\frac{\partial Q}{\partial v^j} = 0 \tag{39}$$

for fields without explicit time-dependence.

For constant uniform fields one would obtain as solutions of (39) the following set of coordinates for the reduced space

$$\begin{aligned} \overrightarrow{Q}^{(0)} &= \overrightarrow{x} + \tfrac{mc}{e\left|\overrightarrow{B}\right|^2}\left(\overrightarrow{v}_{\perp} - \overrightarrow{u}_{\perp}\right) \times \overrightarrow{B} \\ Q_v^{(0)} &= \hat{b} \cdot \overrightarrow{v} \end{aligned} \tag{40}$$

Corrections to these coordinates for the general case arise from the deviation from uniformity in the fields. Therefore an iteration scheme may be devised to construct these corrections with convergence dependent on the fast approach to zero of the higher space derivatives of the fields. Write (39) as

$$\gamma_i \frac{\partial}{\partial z_i} Q = 0 \tag{41}$$

with $\gamma = \left(\vec{\gamma}_x, \vec{\gamma}_v \right)$ and $z = \left(\vec{x}, \vec{v} \right)$

Consider now an arbitrary unit vector $\hat{a}$ transverse to $\hat{b}$. We may take $\hat{a} = \frac{c \times \hat{b}}{\left| c \times \hat{b} \right|}$, c being a fixed vector in space, non collinear with $\hat{b}$. Then define

$$\theta_0 = \frac{mc}{e\,|B|} \cos^{-1} \left(\frac{\hat{a} \cdot \left(\vec{v}_\perp - \vec{u}_\perp \right)}{\left| \vec{v}_\perp - \vec{u}_\perp \right|} \right) \tag{42}$$

θ_0 is the angle variable conjugate to M in the case of uniform fields. Indeed, in this case $\gamma_i \frac{\partial}{\partial z_i} \theta_0$ reduces to

$$\left(\vec{v}_\perp - \vec{u}_\perp \right)_j \left\{ \frac{\partial}{\partial x_j} + \frac{e}{mc} B_{kj} \frac{\partial}{\partial v_k} \right\} \theta_0 = 1 \tag{43}$$

We now write (41) as

$$\gamma_i \frac{\partial}{\partial z_i} \left(Q^{(0)} + Q^{(1)} + \cdots \right) = 0 \tag{44}$$

or

$$\gamma_i \frac{\partial}{\partial z_i} \left(Q^{(1)} + \cdots \right) = -\gamma_i \frac{\partial}{\partial z_i} Q^{(0)} \tag{45}$$

Putting

$$Q^{(1)} = -\theta_0 \gamma_i \frac{\partial}{\partial z_i} Q^{(0)} \tag{46}$$

one cancels the lowest order (in the field derivatives) terms in the right-hand side of (45). Iterating the procedure one finally obtains

$$Q = \frac{1}{1 + \theta_0 \gamma_i \frac{\partial}{\partial z_i}} Q^{(0)} \tag{47}$$

As stated, convergence of the formal series (47) will rely on the fast convergence to zero of higher order space derivatives of the fields. These reduced space coordinates are the variables that should enter into the gyrokinetics reduced Vlasov equation.

In conclusion: By proving the existence of a vector function $\overrightarrow{u}_{\perp}$ satisfying Eq.(22) we have somehow reversed the usual approach to gyrokinetics. Instead of the (still open) question of asymptotic gyroangle independence in the perturbative approach, we have shown the existence of an exact reduction and, whenever approximations are needed, they will focus on truncations of the exact $\overrightarrow{u}_{\perp}$ and the corresponding approximations for the set of variables $\left(\overrightarrow{Q}, Q_v\right)$ in the reduced space.

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